Arithmetic Algorithms: Discrete Fourier Transform

The present tour studies values of a given function in a fixed finite set of sample points and shows that, with respect to such points, the function can be replaced by a finite linear combination of sinusoidal functions.

More precisely, we are given an interval, e.g., the interval $I = [0, 2\pi]$ (since 2π is the period of the function $f(x) = \sin x$), and N points $x_0, x_1, \ldots, x_{N-1}$ regularly spread out over the interval, i.e., $x_\ell = 2\pi\ell/N$ for $\ell = 0, 1, \ldots, N-1$, and some real valued function φ, defined on the interval I.

Assuming that N is even, there is the unique collection of real numbers b_k, $k = 1, \ldots, N/2-1$, c_k, $k = 1, \ldots, N/2$, and d such that

$$\varphi(x_\ell) = \sum_{k=1}^{N/2-1} b_k \sin(kx_\ell) + \sum_{k=1}^{N/2-1} c_k \cos(kx_\ell) + d \qquad \text{for each } \ell = 0, \ldots, N-1. \tag{1}$$

A more general and more mathematically elegant (but equivalent) form of the statement is that if φ is a *complex* function, defined on I, then there are complex numbers $a_0, a_1, \ldots, a_{N-1}$ such that

$$\varphi(x_\ell) = \sum_{k=0}^{N-1} a_k \omega^k \ell \qquad \text{for each } \ell = 0, \ldots, N-1, \tag{2}$$

where ω is a complex number such that $\omega^N = 1$, and the values $\omega^0, \omega^1, \omega^2, \ldots \omega^{N-1}$ are mutually different (note that $\omega^0 = 1$ and $\omega^1 = \omega$). An example of such ω is $e^{2\pi i/N}$ (or $e^{-2\pi i/N}$).

The statements will be proved in the Appendix; the aim of this tour is to show the motivation and use of such statements.

Function Drawing

The first scene just shows how to draw a function in the upper rectangle. The absolute values are not important, but you can assume that the width of the rectangle is 2π, and the vertical distance from the line in the middle of the rectangle (representing the value 0) both to the top and the bottom of the rectangle is 1.

Push the button of the mouse when the cursor is inside the upper rectangle, and drag. Play with the mouse to get familiar with the function drawing.

The functions you can draw in this way are usually not very smooth, but quite sufficient for our purposes.

Sinusoids

The function drawn in the previous scene and many function obtained in an alternative way are not sufficiently smooth and do not have a clear structure. The main goal of the Fourier analysis is to replace such functions (at least in a limited sense) by a linear combination of simple and smooth functions. The present scene shows one class of such simple functions - sinusoids. Catch the top of the red rectangle, representing the *amplitude* of the sinusoid, by the mouse, and move it up-and-down. The amplitude of the sinusoid changes correspondingly.

At the botton of the lower rectangle, you can see small squares (or rectangles for some window aspects), one of them is checked (by a cross). Try to cross other squares (which unchecks the previously checked square), and manipulate with the red, green or blue column that appears.

Checking the square labeled "C" shows a *constant* function, the amplitude rectangle is blue. Use the reddish squares left to the C-square to show sinusoids - the box labeled by k shows the function $a \sin kx$, where a is given by the height of the red amplitude rectangle, i.e., a sinusoid of the period that fits k times to the main rectangle.

Checking the greenish squares right to the C-square shows cosinusoids (but note that a cosinusoid is just a sinusoid shifted left by one quarter of its period). The amplitude rectangle is green, the label of the box tells how many times the period is shorter than 2π.

It is also possible to change the number of boxes, using a choice in the control bar. For this tour, any even number would be acceptable, but the fastest know algorithm for finding numbers a_k, b_k, and c, called Fast Fourier Transform (FFT - explained in another tour of Algovision), requires N to be the integer power of 2, and hence the choice of N is limited to powers of 2.

Sinusoid Combinations

The present scene is very similar to the previous one, but one can check more squares in the lower rectangle, and set the corresponding amplitudes using the mouse. The red function that appears in the upper rectangle is a linear combination of the selected (co)sinusoids that were multiplied by the selected amplitudes.

Play with the scene to see how variable are the functions we can get in this way.

Function Sampling

When studying some physical or other value that changes in time, we are not (and could not be) interested in values at *all* time points - there is an infinite number of them. Most often, we are interested in values in a finite set of points that are regularily spread over the interval. In this and the remaining scenes, such time points are represented by blue vertical line segments in the upper rectangle.

Draw a function (a black plot) in the same way as in the scene "Function Drawing". Small black circles given by intersections of blue sample lines with the black graph of the function are sample points that have been measured; other values of the black function are not interesting for us. This is why we speak about *Discrete* Fourier analysis.

Note that, after sampling, the values of the black function represent a *vector* - a sequence of numbers of a fixed length.

The choice in the control bar can be used to modify the number of sample time lines.

Spectral Analysis

This is the key scene of the DFT tour. Draw any function in the upper rectangle. The program automatically computes the amplitudes of sinusoids, cosinusoids, and the constant (shown in the lower rectangle) so that the their linear combination (obtained in a way explained is the scene "Sinusoid Combinations") exactly matches the values of the original black function in the sample times.

The collection of amplitudes is called a *spectrum* of the original function.

If the black function is smooth, the red function is almost identical to it. Try to draw a function that has at least one (or more) really big jumps of the value from one sample line to the next one. In such a case the red function might be substantially different *between* the sample lines, but there is no way to make it different at sample times.

You can also change the value of N - this changes both the sample lines in the upper rectangle (there are always N lines) and the number of check squares in the lower rectangle - we have one square corresponding to a constant, $N/2-1$ squares for sinusoids and $N/2$ squares for cosinusoids, altogether N boxes, i.e., N degrees of freedom for the choice of amplitudes, which, as we will show in Appendix, is exactly what we need to match values in N sample times.

Now I can also explain a strange asymmetry between sinusoids ($N/2-1$ frequencies) and cosinusoids ($N/2$ frequencies): the function $\sin(Nx/2)$, the first sinusoid not used, has the value 0 in *all* sample points, and hence it would be of no help in matching the black function in the sample points:

$$\text{For } k = 0, \ldots, N-1 \quad \text{if } x = 2\pi k/N, \text{ then } \sin(Nx/2) = \sin \pi k = 0.$$

Spectrum Search

This scene is very useful for understanding Discrete Fourier Transform. It is very similar to the previous one, but it is *you* who has to estimate the amplitudes of the harmonic functions. Of course, the task is pretty difficult, but you should be able at least to roughly estimate amplitudes for basic low-frequency functions (e.g., whether the amplitude is positive or negative or close to zero, whether the amplitude is large or small, etc.). For any particular sinusoid or cosinusoid (or the constant), change its amplitude and try to find the best match.

After preparing your estimation for a particular sin/cos function, *uncheck* the corresponding check square below the amplitude bar. The system replaces your estimation by the correct value and you can see how precise your estimation was. (The square can be re-checked, the amplitude changed, unchecked ...).

Spectral Compression

The last scene shows one important application of DFT - a data compression. Draw a black function in the upper rectangle. Similarly like in the Spectral Analysis scene, the amplitudes in the lower triangles are correctly set, but could not be changed, even though the check squares appear.

If a particular check square is unchecked, the corresponding amplitude is *not taken into account* when computing the red function.

Start to uncheck the sin/cos/const checkboxes, starting with those with high labels. What you can see is that the red function does not any more match exactly the original black function in sample points, but for a long time it looks very similar to the black one. If, e.g., the black function represents a recorded sound, "cutting off" the amplitudes of the high harmonic functions gives a tone with a very similar "color".

Fourier analysis is a basis for JPEG image compression - very roughly speaking, the digitalized image is decomposed into small square areas of pixels, values of pixels in each area are considered as a vector, the spectrum of the vector is computed, its high order components are erased or reduced, and the result is stored. When the image is being restored, it is computed from the reduced spectrum. The experience says that such an image looks very similar as the original one.

Appendix 1

This part of the text does not describe a scene of the tour, but brings more mathematical description of Discrete Fourier Transform.

It is known that for any complex x,

$$e^{ix} = \cos x + i \sin x, \tag{3}$$

where i is the complex unit ($i = \sqrt{-1}$).

The easiest proof comes from the theory of Taylor series: We know that

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

which gives

$$e^{ix} = 1 + ix - \frac{x^2}{2} - i\frac{x^3}{3!} + \frac{x^4}{4!} + \cdots,$$

and

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \cdots$$

which together gives (3).

Let us first clarify the relation of equations (1) and (2). Assume certain complex-valued function φ, defined at the interval $[0, 2\pi]$, and an even number N. Consider its values in points $x_\ell = 2\pi\ell/N$, where $\ell = 0, \ldots, N-1$. The values will be denoted as $A_\ell = \varphi(x_\ell) = \varphi(2\pi\ell/N)$.

Denote $\omega = e^{2\pi i/N}$. We have

$$\omega^N = \left(e^{2\pi i/N}\right)^N = e^{N\cdot 2\pi i/N} = e^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1 + i\cdot 0 = 1.$$

Moreover numbers $1, \omega, \omega^2, \ldots, \omega^{N-1}$ are different, because for any two different integers r, s, such that $0 \le r, s < N$ it is

$$\frac{\omega^r}{\omega^s} = \omega^{r-s} = \exp\left(\frac{2\pi(r-s)}{N}\right) = \cos 2\pi\xi + i \sin 2\pi\xi, \text{ where } \xi = (r-s)/N,$$

and, according to our assumption, $0 < |r-s| < N$, and hence $0 < \xi < 1$, which implies that $\cos 2\pi\xi \neq 1$, and consequently $\omega^r \neq \omega^s$.

The equation (3) immediately implies

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \tag{4}$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}. \tag{5}$$

Substituting (4) and (5) into (1) (the equation (1) has been given at the beginning of this text), we immediately see that that we can write the value of φ in all sample points x_ℓ as a linear combination of functions e^{kx} for $k = -N/2, \ldots, N/2$, i.e. there are numbers a_k for $k = -N/2, \ldots, N/2$ such that

$$\varphi(x_\ell) = \sum_{k=-N/2}^{N/2} a_k \omega^{k\ell} \qquad \text{for } \ell = 0, \ldots, N-1, \tag{6}$$

On the other hand, if we know coefficients a_k to represent the values of φ in the sample points according to (6), then a substitution using (3) gives us immediately a representation of φ according to (1) (note that for $k = 0$, $\omega^{k\ell}$ is a constant function, equal to 1 everywhere).

We will further modify (6): for any integers k, ℓ,

$$\omega^{(k+N)\ell} = \omega^{k\ell}\omega^{N\ell} = \omega^{k\ell}\left(\omega^N\right)\ell = \omega^{k\ell}1^\ell = \omega^{k\ell},$$

because $\omega^N = 1$.

This implies that $\omega^{-(N/2)\ell} = \omega^{(N/2)\ell}$ and the corresponding terms in (6) could be joined, and, moreover, $\omega^{k\ell}$, $k = -N/2+1, \ldots, -1$ in (6) could be replaced by $\omega^{k\ell}$, $k = N/2+1, \ldots, N-1$, which transforms (6) into

$$\varphi(x_\ell) = \sum_{k=0}^{N-1} a_k \omega^{k\ell} \qquad \text{for } \ell = 0, \ldots, N-1, \tag{7}$$

where $x_\ell = 2\pi\ell/N$ for $\ell = 0, \ldots, N-1$.

Appendix 2

In the second appendix, we will show that, given an arbitrary complex-valued function defined on the interval $[0, 2\pi]$, there are *unique* numbers a_k, $k = 0, \ldots, N-1$ such that (7) holds. The proof is based on elementary Linear Algebra.

Consider the vectors $\mathbf{v}_k$, $k = 0, \ldots, N-1$ in the N-dimensional complex Euclidean vector space of the dimension N that are defined as follows:

$$\mathbf{v}_\ell = (\omega^{0\cdot\ell}, \ldots, \omega^{(N-1)\cdot\ell}.$$

(7) can be reformulated as a statement that the vector $(\varphi(x_0), \ldots.\varphi(x_{N-1}))$ is a *unique* linear combination of vectors $\mathbf{v}_0, \ldots, \mathbf{v}_{N-1}$. Since the vectors represent N vectors in the N-dimensional vector space, it is necessary and sufficient to prove that they are linearly independent, and hence they form a *basis* of the vector space.

No one of the vectors is the null vector, because the first element of each of them is $\omega^{0\cdot k} = 1$. We will even prove that the vectors are mutually orthogonal, and hence they form an orthogonal basis. The orthogonality of $\mathbf{v}_r$ and $\mathbf{v}_s$ means that their scalar product is 0:

$$\mathbf{v}_r \cdot \mathbf{v}_s = \sum_{k=0}^{N-1} \omega^{kr}\overline{\omega_{ks}} = \sum_{k=0}^{N-1} \omega^{kr}\omega_{-ks} = \sum_{k=0}^{N-1} \omega^{kr-ks} = \sum_{k=0}^{N-1} \omega^{k(r-s)}.$$

Assume that r and s are two different integers in the range $0 \le r, s < N$, and denote $q = \omega^{r-s}$. Thus, according to out assumption, $q \neq 1$, and the scalar product of $\mathbf{v}_r$ and $\mathbf{v}_s$ is equal to

$$1 + q + q^2 + \cdots + q^{N-1} = \frac{q^N - 1}{q - 1} = 0,$$

because $(q-1) \neq 0$ and

$$q^N = \left(\omega^{r-s}\right)^N = \left(\omega^N\right)^{r-s} = 1^{r-s} = 1.$$